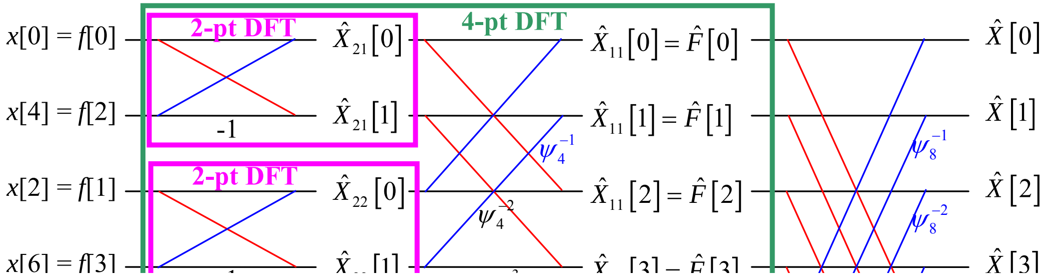

ECS455: Chapter 5

OFDM

5.3 Implementation: DFT and FFT

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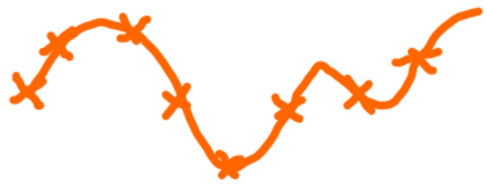
Tuesday	**9:30-10:30**
Tuesday	**13:30-14:30**
Thursday	**13:30-14:30**

Discrete Fourier Transform (DFT)

Transmitter produces

$$s(t) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} S_k \exp\left(j \frac{2\pi k}{T_s} t \right), \quad 0 \le t < T_s$$



Sample the signal **in time domain every T_s / N** gives

sampling rate $= \frac{1}{T_s/N} = \frac{N}{T_s}$

$$s[n] = s\left(n \frac{T_s}{N} \right) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} S_k \exp\left(j \frac{2\pi k}{\cancel{T_s}} n \frac{\cancel{T_s}}{N} \right)$$

$s(t)\Big|_{t = n\frac{T_s}{N}}$

$$= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} S_k \exp\left(j \frac{2\pi k n}{N} \right) = \sqrt{N}\ \cancel{\text{I}}^{\text{F}}\text{DFT}\{S\}[n]$$

where $\text{IDFT}\{\mathbf{S}\}[n] = \frac{1}{N} \sum_{k=0}^{N-1} S_k \exp\left(j \frac{2\pi k n}{N} \right)$

$\mathbf{S} = \begin{pmatrix} S_0 & S_1 & \cdots & S_{N-1} \end{pmatrix}^T$

We can implement OFDM in the discrete domain!

Discrete Fourier Transform (DFT)

Here, we work with N-point signals (finite-length sequence (vector) of length N) in both time and frequency domain.

$$\mathbf{x} = \begin{pmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{pmatrix} \longrightarrow \boxed{\text{DFT}} \longrightarrow \mathbf{X} = \begin{pmatrix} X[0] \\ X[1] \\ \vdots \\ X[N-1] \end{pmatrix}$$

$$\mathbf{X} = \text{DFT}\{\mathbf{x}\} = \mathbf{\Psi}_N \mathbf{x}$$

To simplify the notation, we define the **DFT matrix $\mathbf{\Psi}_N$**.

$$(\mathbf{\Psi}_N)^{-1}\mathbf{X} = \text{IDFT}\{\mathbf{X}\} = \mathbf{x} \underset{\text{IDFT}}{\overset{\text{DFT}}{\rightleftharpoons}} \mathbf{X} = \text{DFT}\{\mathbf{x}\} = \mathbf{\Psi}_N \mathbf{x}$$

DFT matrix $\mathbf{\Psi}_N$

$$\mathbf{\Psi}_N = \begin{array}{c} 1 \\ 2 \\ 1 \\ 1 \\ 1 \end{array} \begin{bmatrix} \boxed{\begin{matrix} 1 & 1 \\ 1 & \psi_N^{-1} \end{matrix}} & \begin{matrix} 1 \\ \psi_N^{-2} \end{matrix} & \begin{matrix} \cdots \\ \cdots \end{matrix} & \begin{matrix} 1 \\ \psi_N^{-(N-1)} \end{matrix} \\ 1 \quad \psi_N^{-2} & \psi_N^{-4} & \cdots & \psi_N^{-2(N-1)} \\ \vdots \quad \vdots & \vdots & \ddots & \vdots \\ 1 \quad \psi_N^{-(N-1)} & \psi_N^{-2(N-1)} & \cdots & \psi_N^{-(N-1)(N-1)} \end{bmatrix}$$

Element on the pth row and qth column is given by

$$\psi_N^{-(p-1)(q-1)} \text{ where } \psi_N = e^{j\frac{2\pi}{N}}$$

Note that the "-1" are there because we start from row 1 and column 1 (not from row 0 and column 0).

$\psi_2 = e^{j\frac{2\pi}{2}} = e^{j\pi}$

$= -1$

$\Psi_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

Key Property: $\boxed{\mathbf{\Psi}_N^{-1} = \frac{1}{N}\mathbf{\Psi}_N^{*}}$ complex conjugate $\Rightarrow \frac{1}{\sqrt{N}}\mathbf{\Psi}_N$ is a unitary matrix

$$\frac{1}{N}\mathbf{\Psi}_N^{*}\mathbf{X} = \left(\mathbf{\Psi}_N\right)^{-1}\mathbf{X} = \mathrm{IDFT}\left\{\mathbf{X}\right\} = \mathbf{x} \underset{\text{IDFT}}{\overset{\text{DFT}}{\rightleftharpoons}} \mathbf{X} = \mathrm{DFT}\left\{\mathbf{x}\right\} = \mathbf{\Psi}_N\mathbf{x}$$

$\vec{x} = \begin{bmatrix} 2 \\ 5 \end{bmatrix} \quad \vec{X} = \mathrm{DFT}(\vec{x}) = \Psi_2 \vec{x} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 7 \\ -3 \end{bmatrix}$

$$\mathrm{IDFT}(\vec{X}) = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 7 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

Discrete Fourier Transform (DFT)

Matrix form:

$$\boxed{\frac{1}{N}\mathbf{\Psi}_N^*\mathbf{X} = \mathrm{IDFT}\{\mathbf{X}\} = \mathbf{x} \underset{\text{IDFT}}{\overset{\text{DFT}}{\rightleftharpoons}} \mathbf{X} = \mathrm{DFT}\{\mathbf{x}\} = \mathbf{\Psi}_N\mathbf{x}}$$

Pointwise form:

$$\frac{1}{N}\sum_{k=0}^{N-1} X[k]\psi_N^{nk} = \underset{0\le n<N}{x[n]} \underset{\text{IDFT}}{\overset{\text{DFT}}{\rightleftharpoons}} \underset{0\le k<N}{X[k]} = \sum_{n=0}^{N-1} x[n]\psi_N^{-nk}$$

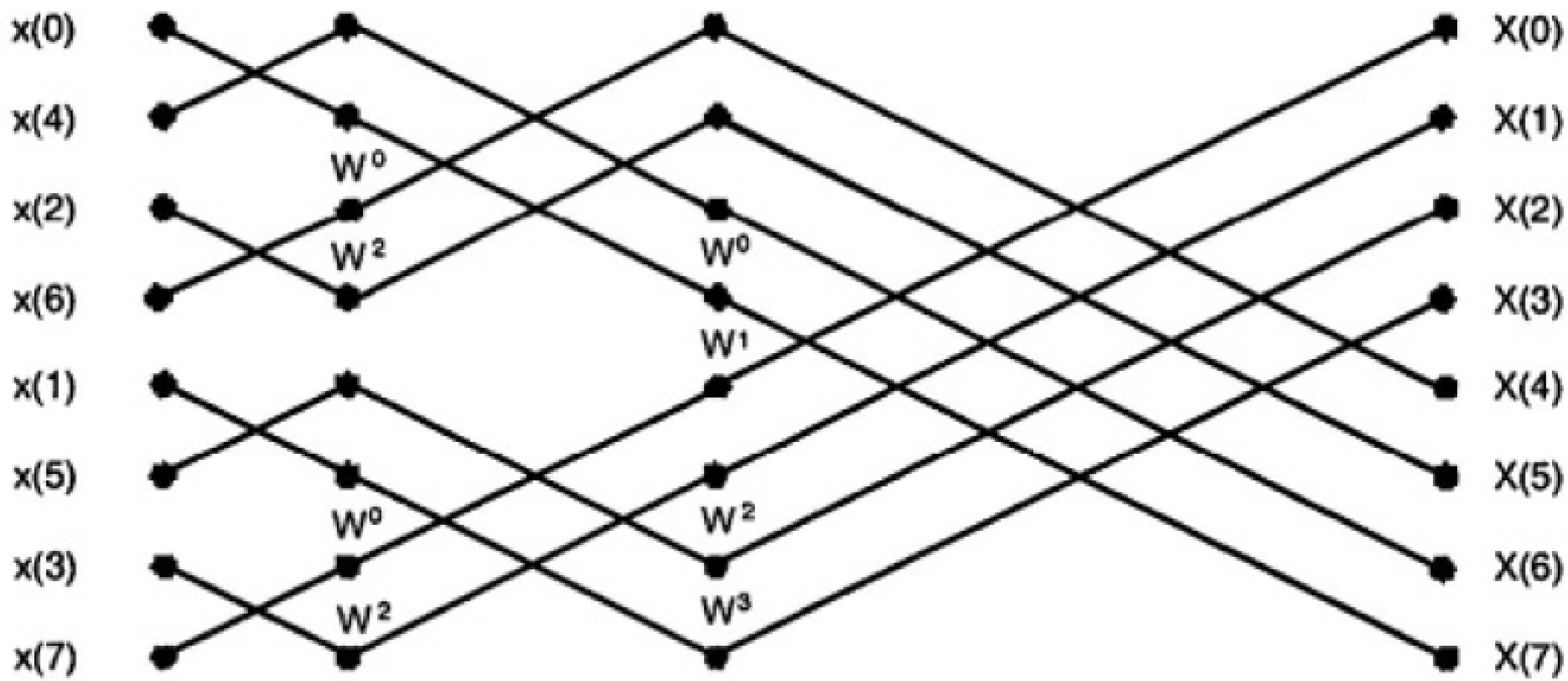
or, equivalently,

$$\boxed{\frac{1}{N}\sum_{n=0}^{N-1} X[k]e^{jnk\frac{2\pi}{N}} = \underset{0\le n<N}{x[n]} \underset{\text{IDFT}}{\overset{\text{DFT}}{\rightleftharpoons}} \underset{0\le k<N}{X[k]} = \sum_{n=0}^{N-1} x[n]e^{-jnk\frac{2\pi}{N}}}$$

Comparison with Fourier transform

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} x(f) = \int_{-\infty}^{\infty} X(t)e^{-j2\pi ft}dt$$

Efficient Implementation: (I)FFT

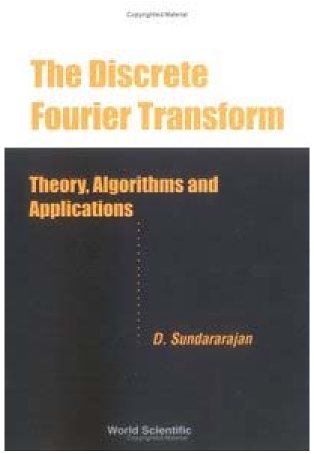


[Bahai, 2002, Fig. 2.9]

An N-point FFT requires only on the order of $N\log N$ multiplications, rather than N^2 as in a straightforward computation.

FFT

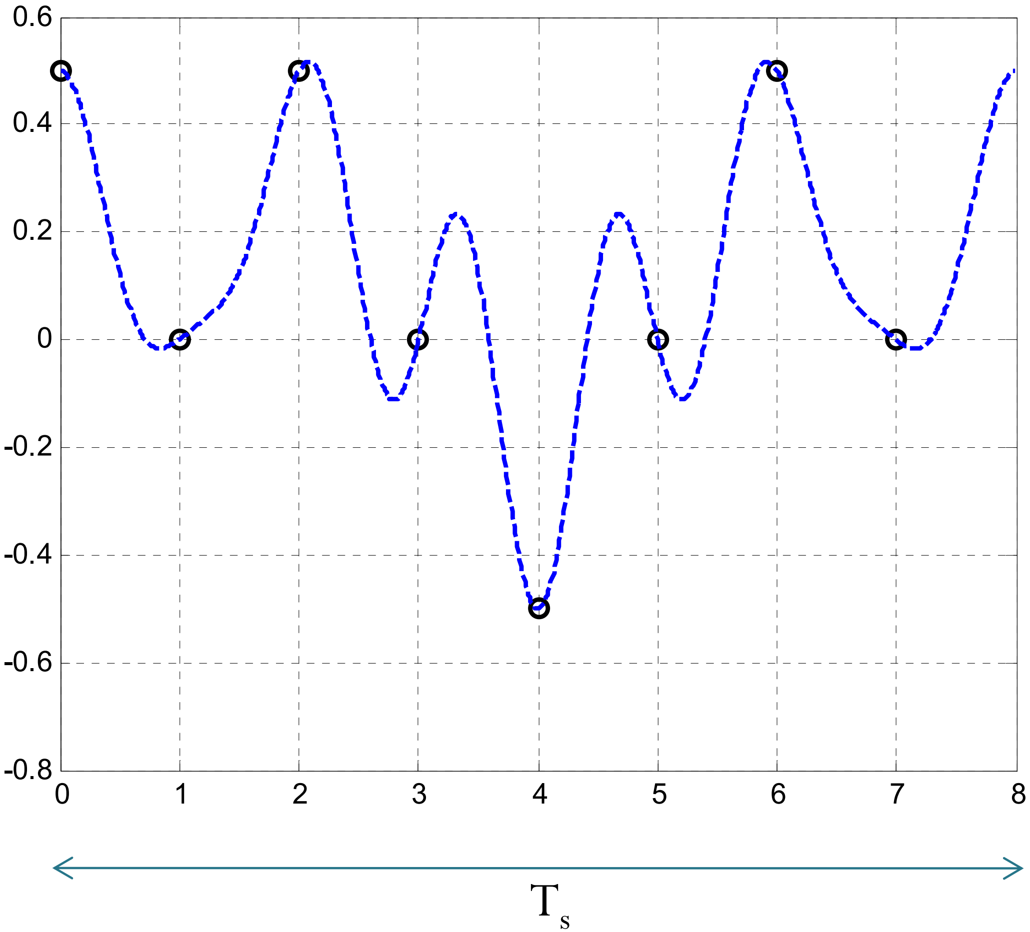
- The history of the FFT is complicated.
- As with many discoveries and inventions, it arrived before the (computer) world was ready for it.
- Usually done with *N* a power of two.
  - Very efficient in terms of computing time
  - Ideally suited to the binary arithmetic of digital computers.
  - Ex: From the implementation point of view it is better to have, for example, a FFT size of 1024 even if only 600 outputs are used than try to have another length for FFT between 600 and 1024.



References: E. Oran Brigham, *The Fast Fourier Transform*, Prentice-Hall, 1974.

DFT Samples

$$s(t) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} S_k \exp\left(j \frac{2\pi k t}{T_s} \right), \quad 0 \le t \le T_s$$



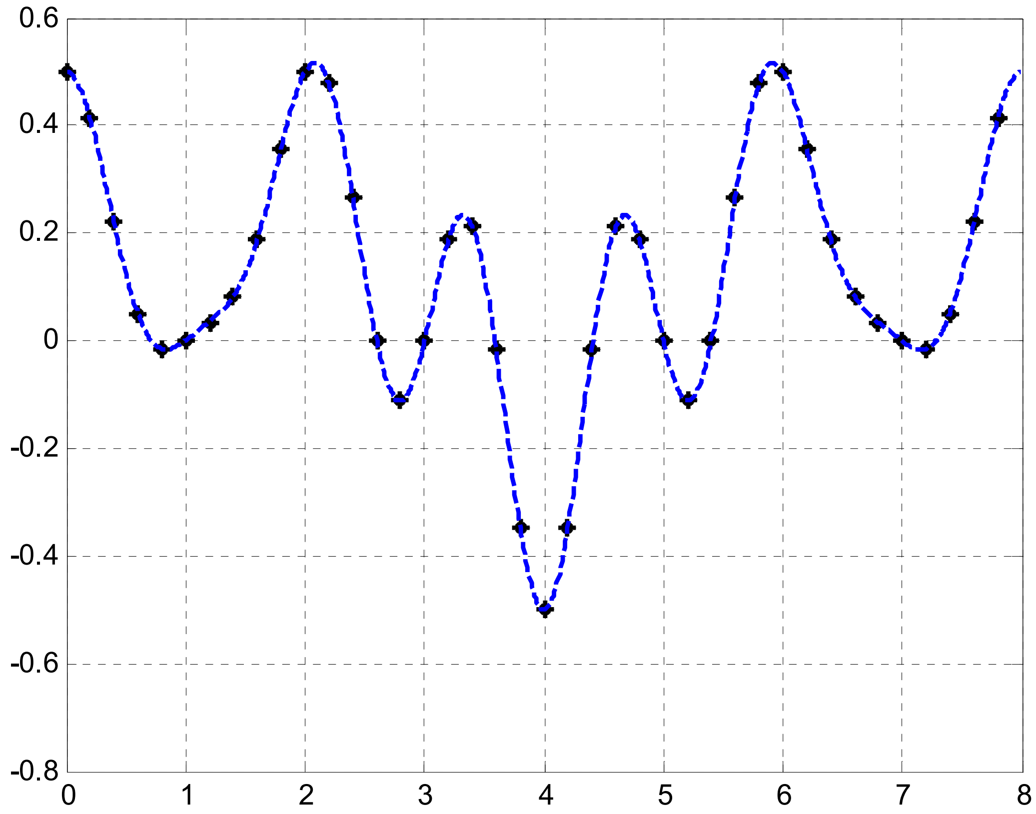
$$s[n] = s\left(n \frac{T_s}{N} \right)$$

$$= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} S_k \exp\left(j \frac{2\pi k n}{N} \right)$$

$$= \sqrt{N}\ \mathrm{IDFT}\{S\}[n]$$

$$0 \le n < N$$

Oversampling



Oversampling (2)

- Increase the number of sample points from N to LN on the interval $[0, T_s]$.
- L is called the **over-sampling factor**.

$$s[n] = s\left(n\frac{T_s}{N}\right) \qquad 0 \le n < N$$

$$\Rightarrow$$

$$s^{(L)}[n] = s\left(n\frac{T_s}{LN}\right) \qquad 0 \le n < LN$$

$$s^{(L)}[n] = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} S_k \exp\left(j\frac{2\pi k}{\cancel{T_s}} n \frac{\cancel{T_s}}{LN}\right) = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} S_k \exp\left(j\frac{2\pi kn}{LN}\right)$$

$$= \frac{1}{\sqrt{N}} LN\left(\frac{1}{LN}\sum_{k=0}^{N-1} S_k \exp\left(j\frac{2\pi kn}{LN}\right)\right)$$

$$= L\sqrt{N}\left(\frac{1}{LN}\left(\sum_{k=0}^{N-1} S_k \exp\left(j\frac{2\pi kn}{LN}\right) + \sum_{k=N}^{NL-1} 0 \exp\left(j\frac{2\pi kn}{LN}\right)\right)\right)$$

$$= L\sqrt{N}\left(\frac{1}{LN}\sum_{k=0}^{NL-1} \tilde{S}_k \exp\left(j\frac{2\pi kn}{LN}\right)\right) = L\sqrt{N}\ \mathrm{IDFT}\{\tilde{S}\}[n]$$

Zero padding:

$$\tilde{S}_k = \begin{cases} S_k, & 0 \le k < N \\ 0, & N \le k < LN \end{cases}$$

Scaling

Oversampling: Summary

N points

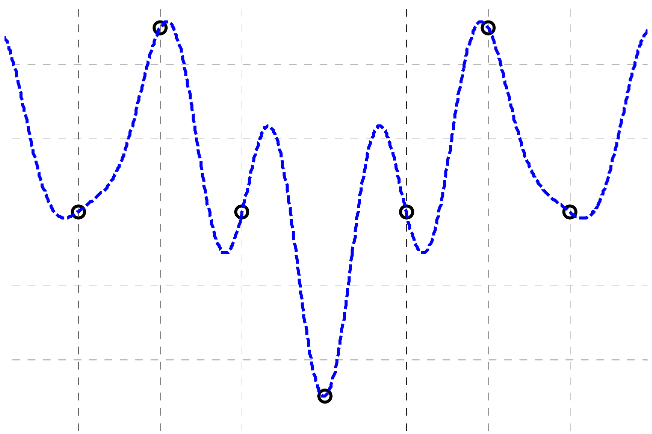
$$s[n] = s\left(n\frac{T_s}{N}\right) = \sqrt{N}\ \mathrm{IDFT}\{\mathbf{S}\}[n]$$
$$0 \le n < N$$

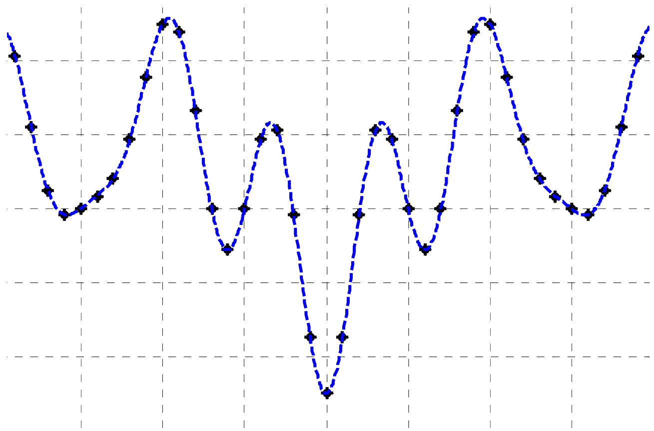
LN points

$$s^{(L)}[n] = s\left(n\frac{T_s}{LN}\right) = L\sqrt{N}\ \mathrm{IDFT}\{\tilde{\mathbf{S}}\}[n]$$
$$0 \le n < LN$$

Zero padding:

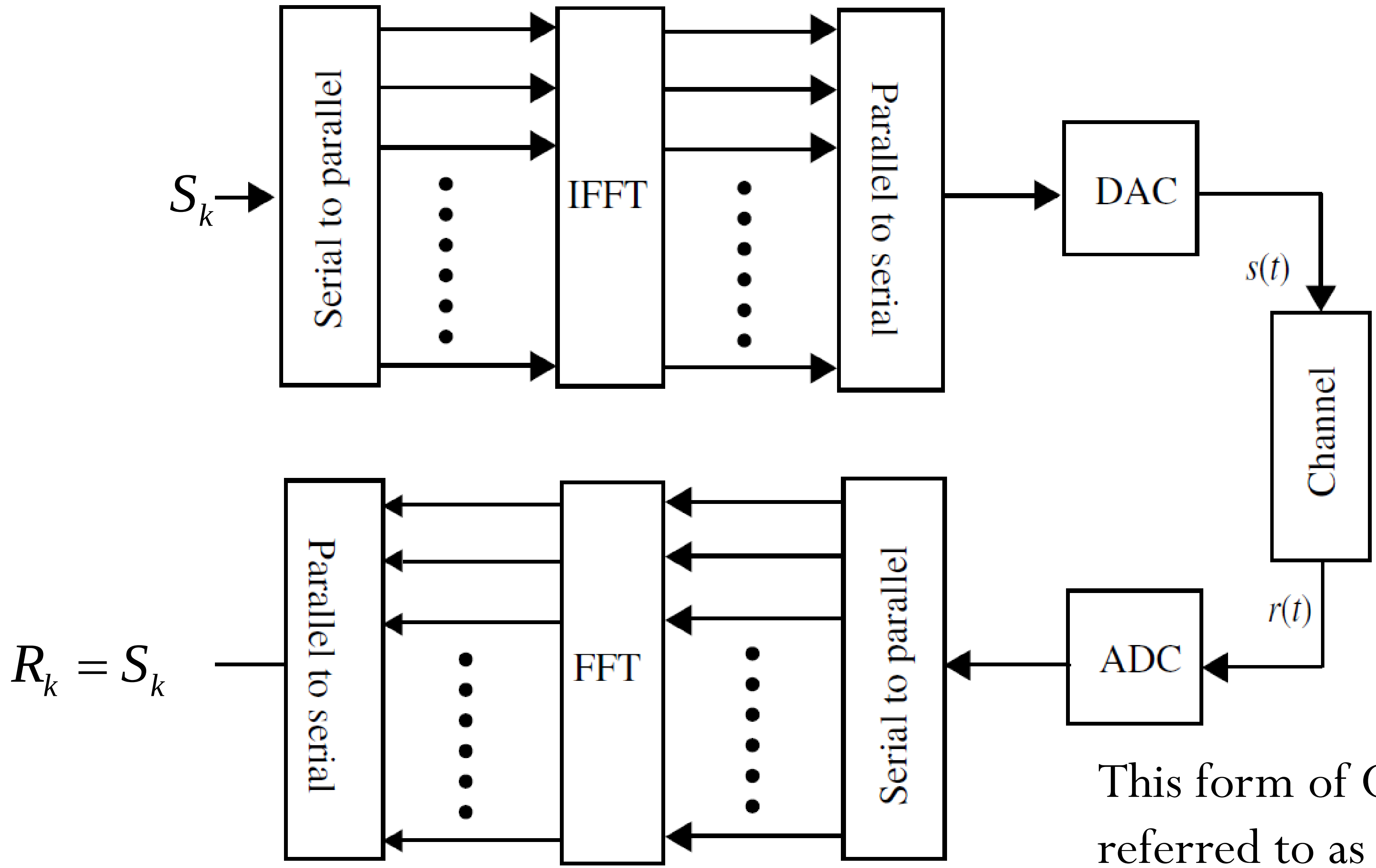
$$\tilde{S}_k = \begin{cases} S_k, & 0 \le k < N \\ 0, & N \le k < LN \end{cases}$$





OFDM implementation by IFFT/FFT

$$s^{(L)}[n] = s\left(n\frac{T_s}{LN}\right) = L\sqrt{N}\,\mathrm{IFFT}^{(L)}\{\tilde{\mathbf{S}}\}[n]$$



$$\mathrm{r}[\mathrm{n}] = \sqrt{\mathrm{N}}\,\mathrm{IFFT}\{\mathbf{S}\}[\mathrm{n}]$$

This form of OFDM is often referred to as **Discrete Multi-Tone (DMT)**.

OFDM with Memoryless Channel

$$h(t) = \beta\delta(t)$$

$\left[\text{should be } h(t) = \beta\delta(t-\tau)\right]$

$$r(t) = h(t) * s(t) + w(t) = \beta s(t) + w(t)$$

Additive white Gaussian noise

Sample every T_s/N

$$r[n] = \beta s[n] + w[n]$$

$$s[n] = \sqrt{N}\,\text{IFFT}\{S\}[n]$$

FFT

$$R_k = \frac{1}{\sqrt{N}}\text{FFT}\{\mathbf{r}\}[n] = \beta S_k + \frac{1}{\sqrt{N}}W_k$$

Sub-channel are independent.

(No ICI)

Channel with Finite Memory

Discrete time baseband model:

$$y[n] = \{h * s\}[n] + w[n] = \sum_{m=0}^{\nu} h[m]s[n-m] + w[n]$$

[Tse Viswanath, 2005, Sec. 2.2.3]

where $h[n] = 0 \text{ for } n < 0 \text{ and } n > \nu$

$$w[n] \overset{i.i.d.}{\sim} \mathcal{CN}(0, N_0)$$

We will assume that $\nu \ll N$

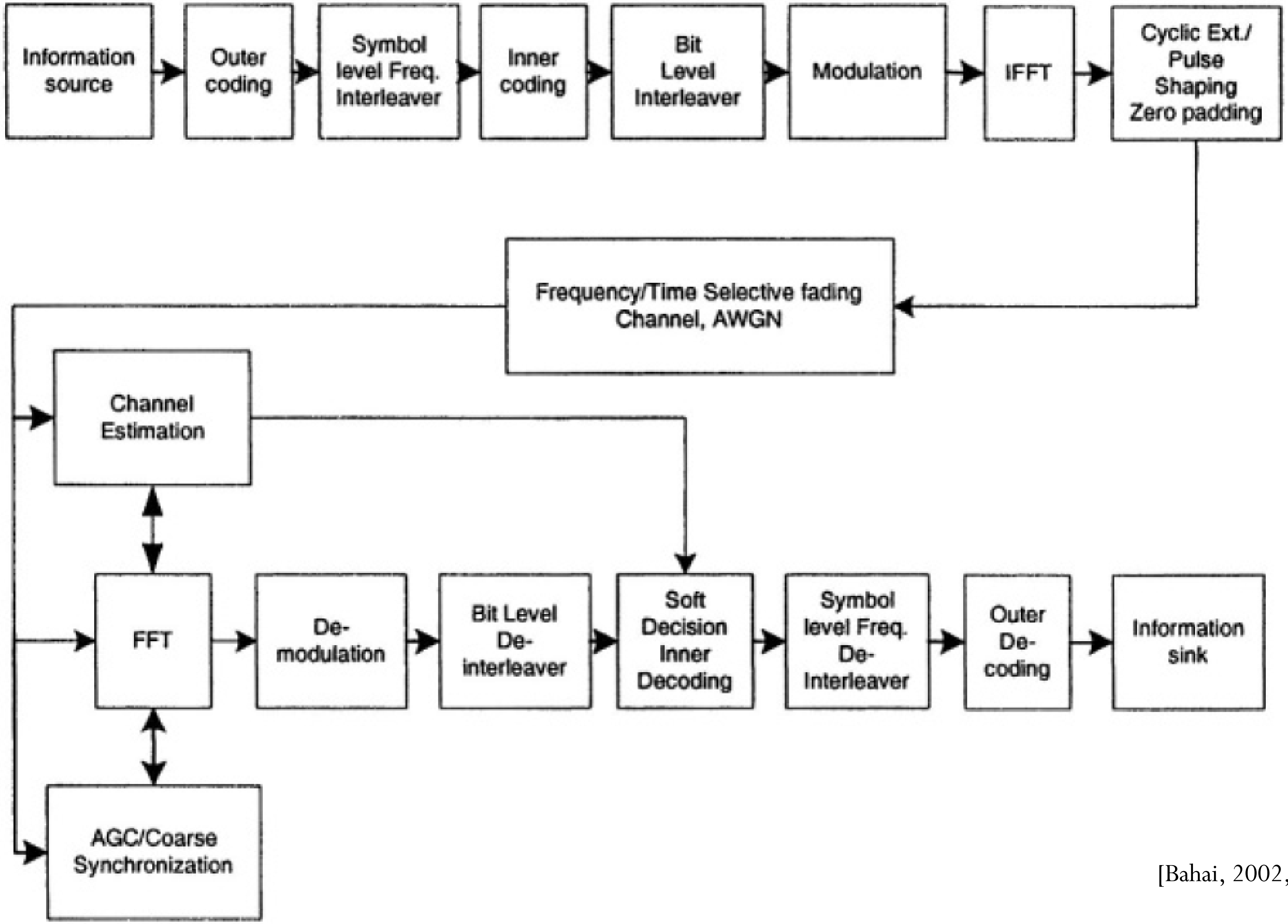
Remarks:

$Z = X + jY$ is a ***complex Gaussian*** if X and Y are jointly Gaussian.

If X, Y is i.i.d. $\mathcal{N}(0, \sigma^2)$, then $Z = X + iY \sim \mathcal{CN}(0, \sigma_Z^2)$ where $\sigma_Z^2 = 2\sigma^2$ with

$$f_Z(z) = f_{X,Y}(\mathrm{Re}\{z\}, \mathrm{Im}\{z\}) = \frac{1}{\pi\sigma_Z^2} e^{-\frac{|z|^2}{\sigma_Z^2}}.$$

OFDM Architecture



[Bahai, 2002, Fig 1.11]